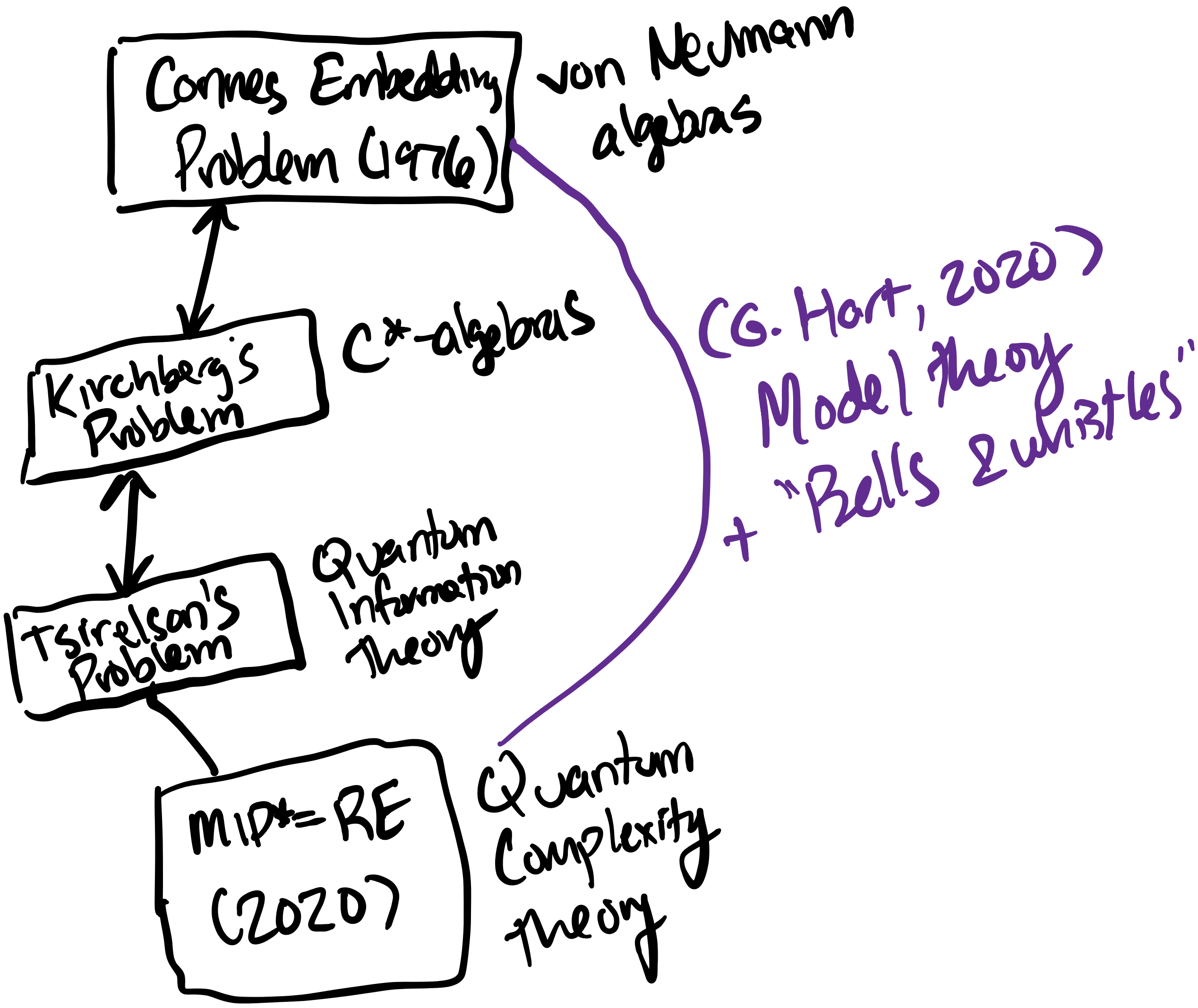
The Connes Embedding Problem, MIP* = RE, and Model Theory

Webpage: https://www.math.uci.edu/~isaac/yonsei.html

Recommended Reading: The Connes Embedding Problem: A Guided Tour arXiv 2109.12682



$$\mathrm{MIP}^* = \mathrm{RE} \Rightarrow \neg \mathrm{CEP}$$

von Neumann algebras

$\mathcal{H}$ complex Hilbert space

For linear $T : \mathcal{H} \to \mathcal{H}$,

$$\|T\| := \sup\{\|Tx\| : x \in \mathcal{H},\ \|x\| \le 1\}$$

operator norm

$$\mathcal{B}(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H} \text{ linear} : \|T\| < \infty\}.$$

$\mathcal{B}(\mathcal{H})$ is a unital *-algebra: algebra over $\mathbb{C}$,
$* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is an involution
($\langle Tx, y\rangle = \langle x, T^* y\rangle$ $\forall x, y \in \mathcal{H}$), $\lambda \in \mathbb{C}$.

$$(T+S)^* = T^* + S^*,\quad (TS)^* = S^*T^*,\quad (\lambda T)^* = \bar{\lambda}\, T^*$$

Operator algebras: "closed" *-subalgebras of $\mathcal{B}(H)$

C*-algebras: *-subalg. of $\mathcal{B}(H)$ closed in topology induced by $\|\cdot\|$.

von Neumann algebras: unital *-subalg of $\mathcal{B}(H)$ closed in weak operator topology (WOT) weakest topology on $\mathcal{B}(H)$ so that $T \mapsto |\langle Tx, y\rangle| : \mathcal{B}(H) \to \mathbb{C}$ is cont. for every $x, y \in H$.

WOT $\subseteq$ norm top, so vNas are C^*-alg. Converse false. vNas are "bigger".

For $X \subseteq \mathcal{B}(H)$, $X' = \{T \in \mathcal{B}(H) : TS = ST \ \forall S \in X\}$

Thm (von Neumann bicommutant theorem)
If A is a unital $*$-subalg of $\mathcal{B}(H)$, then A is a vNa iff $A = A''$.

examples
① $\mathcal{B}(H)$. In particular $M_n(\mathbb{C})$.
② If (X, μ) is a σ-finite measure space, view $L^\infty(X,\mu)$ as a subalgebra of $\mathcal{B}(L^2(X,\mu))$ by $f \mapsto M_f$, $M_f(g) := fg$.
Check: $L^\infty(X,\mu)' = L^\infty(X,\mu)$.
$\therefore L^\infty(X,\mu)$ is an abelian vNa.

Fact: All abelian vNa's are of this kind.
"noncommutative measure theory"

(3) If Γ is a group, let $\ell^2(\Gamma) = \{f : \Gamma \to \mathbb{C} : \sum_{\gamma \in \Gamma} |f(\gamma)|^2 < \infty\}$.

For $\eta \in \Gamma$, get $u_\eta \in \mathcal{B}(\ell^2(\Gamma))$ by

$$(u_\eta(f))(\gamma) := f(\eta^{-1}\gamma) \qquad u_\eta^* = u_\eta^{-1} = u_{\eta^{-1}}$$

unitary operators

$u : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ unitary rep. of Γ
called left-regular rep.
The vNa generated by $u(\Gamma)$ in $\mathcal{B}(\ell^2(\Gamma))$ is denoted $L(\Gamma)$, <u>group vNa</u>.

Def: A <u>trace</u> on the vNa M is a linear functional $\tau : M \to \mathbb{C}$ (think: integration in case $M = L^\infty(X, \mu)$ or <u>normalized</u> trace when $M = M_n(\mathbb{C})$) satisfying:

- τ is <u>positive</u>: $\tau(x^*x) \geq 0$
  typical positive element
- τ is <u>normal</u>: "continuity condition" ← vNa
- τ is <u>tracial</u>: $\tau(xy) = \tau(yx)$
- $\tau(1) = 1$

A **tracial vNa** is a pair (M, τ), M vNa,

τ is a trace on M.

examples

① $(M_n(\mathbb{C})$, normalized trace$)$. But: if $\dim(H)=\infty$, $\mathcal{B}(H)$ does not have a trace.

② $(L^\infty(X,\mu), \int)$ for (X,μ) prob space.

③ $(L(\Gamma), \tau)$, $\tau(x) = \langle x\delta_e, \delta_e \rangle$

For $x = \sum c_\gamma u_\gamma \in \mathbb{C}[\Gamma] \subseteq L(\Gamma)$, $\tau(x) = c_e$.

④ (Main Example!)

$$M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \hookrightarrow M_8(\mathbb{C}) \hookrightarrow \cdots$$

If $A \in M_{2^n}(\mathbb{C})$, view it as an element of $M_{2^{n+1}}(\mathbb{C})$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$.

$*$-algebra embeddings, preserve normalized trace

$M := \bigcup_{n\in\mathbb{N}} M_{2^n}(\mathbb{C})$ $*$-algebra with a "trace."

GNS construction! View M as a $*$-subalg of $\mathcal{B}(L^2(M,\tau))$

The vNa generated by M is called the hyperfinite II_1 factor, $\mathcal{R}$.

The "trace" above gives an actual trace on $\mathcal{R}$

Def If M is a vNa, the center of M is $Z(M) := \{x \in M : xy = yx \ \forall y \in M\} = M \cap M'$

Always: $\mathbb{C} \cdot 1 \subseteq Z(M)$.

M is a factor if $Z(M) = \mathbb{C} \cdot 1$. If a factor has a trace, it is unique!

"Building blocks" of vNa's.

Type classification of factors:

$\underline{I}$	I_n $M_n(\mathbb{C})$	I_∞ $\mathcal{B}(H)$ H inf. dim
$\underline{II}$	II_1	II_∞
$\underline{III}$	$(III_\lambda)_{\lambda \in [0,1]}$	

II_1: admit a trace, but inf. dim.

examples

① $\mathcal{R}$. In fact, $\mathcal{R}$ embeds into any II_1 factor.

② If Γ is a group, then $L(\Gamma)$ is a factor iff Γ is ICC: all nontrivial conjugacy classes are infinite. If Γ is infinite, then

then $L(\Gamma)$ is a II_1 factor.

Thm (Connes) If Γ is amenable, ^{ICC} then $L(\Gamma) \cong \mathcal{R}$.

e.g. $\Gamma = \bigcup_n S_n$.

Famous Open Q: $L(\mathbb{F}_2) \cong L(\mathbb{F}_3)$?

Tracial ultraproduct construction

Fix family $(M_i, \tau_i)_{i \in I}$ of tracial vNas and ultrafilter $\mathcal{U}$ on I.

$$\ell^\infty(M_i) = \{(x_i) \in \prod_{i \in I} M_i : \sup_{i \in I} \|x_i\| < \infty\}.$$

Idea: Define a "trace" on $\ell^\infty(M_i)$ by

$$\tau(x_i) = \lim_{\mathcal{U}} \tau_i(x_i)$$

Issue: $\tau(x_i) = 0$ even if $(x_i) \neq (0, 0, \dots)$

set $C_{\mathcal{U}} := \{(x_i) \in \ell^\infty(M_i) : \lim_{\mathcal{U}} \tau_i(x_i^* x_i) = 0\}$.

set $\prod_{\mathcal{U}} (M_i, \tau_i) = \ell^\infty(M_i) / C_{\mathcal{U}}$ tracial ultraproduct

Fact: This is a vNa with a trace

$$\tau((x_i)_{\mathcal{U}}) := \lim_{\mathcal{U}} \tau_i(x_i).$$

If all $(M_i, \tau_i) = (M, \tau)$, get **tracial ultrapower** $(M, \tau)^{\mathcal{U}}$.

Recall: $\mathcal{R}$ embeds into any II_1 factor.

Connes Embedding Problem If M is a II_1 factor, does M embed into $\mathcal{R}^{\mathcal{U}}$? in a trace-preserving way

Alternation: $M \longleftrightarrow \mathrm{Th}_{\mathcal{U}}\, M_n(\mathbb{C})$? Same Q.

Model Theory Version If M is a II_1 factor, is $\mathrm{Th}_\forall(M) = \mathrm{Th}_\forall(\mathcal{R})$?

RE
recursively enumerable

$p_1(\vec{x}), \ldots, p_m(\vec{x})$ $*$-polynomials in $x_1, \ldots, x_n$
$\vec{a} = a_1, \ldots, a_n$ in II_1 factor M, $\varepsilon > 0$.
Must there be $\vec{b} = b_1, \ldots, b_n$ in $\mathcal{R}$ so that
$|\tau_M(p_i(\vec{a})) - \tau_{\mathcal{R}}(p_i(\vec{b}))| < \varepsilon$?

or in some $M_k(\mathbb{C})$

"Microstates Conjecture"

$\mathcal{R}^{\mathcal{U}} \not\hookrightarrow \prod_{\mathcal{U}} M_n(\mathbb{C})$ (Farah, Hart, Sherman)

has Gamma $\quad$ doesn't have Gamma

$\forall\exists$

$$\prod_{\mathcal{U}} M_n(\mathbb{C}) \hookrightarrow \mathcal{R}^{\mathcal{U}} \xhookrightarrow{\prod_{\mathcal{U}} E_n} \prod_{\mathcal{U}} M_n(\mathbb{C})$$

$$E_n : \mathcal{R} \longrightarrow M_n(\mathbb{C})$$